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ON A DIMENSION FORMULA FOR TWISTED SPHERICAL CONJUGACY CLASSES IN SEMISIMPLE ALGEBRAIC GROUPS

JIANG-HUA LU

Abstract. Let G be a connected semisimple algebraic group over an algebraically closed field of characteristic zero, and let θ be an automorphism of G. We give a characterization of θ-twisted spherical conjugacy classes in G by a formula for their dimensions in terms of certain elements in the Weyl group of G, generalizing a result of N. Cantarini, G. Carnovale, and M. Costantini when θ is the identity automorphism. For G simple and θ an outer automorphism of G, we also classify the Weyl group elements that appear in the dimension formula.

1. Introduction

1.1. **The main results.** Let G be a connected semisimple algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic zero. For an automorphism $\theta \in \mathrm{Aut}(G)$ of G, define the θ-twisted conjugation of G on itself by $g_1 \cdot_\theta g = g_1 g \theta(g_1)^{-1}$ for $g_1, g \in G$, and call its orbits the θ-twisted conjugacy classes in G. A θ-twisted conjugacy class C in G is said to be spherical if a Borel subgroup of G has an open orbit in C.

Fix a Borel subgroup B of G and a maximal torus $H \subset B$, and let $\mathrm{Aut}'(G) = \{\theta \in \mathrm{Aut}(G) : \theta(B) = B, \theta(H) = H\}$. Throughout the paper, we assume that $\theta \in \mathrm{Aut}'(G)$ (see Remark 1.2).

Let $W = N_G(H)/H$ be the Weyl group, where $N_G(H)$ is the stabilizer subgroup of H in G, and let l be the length function on W. For $w \in W$, denote by $\mathrm{rk}(1 - w\theta)$ the rank of the linear operator $1 - w\theta$ on the Lie algebra $\mathfrak{h}$ of H. For a θ-twisted conjugacy class C in G, let m_C be the unique element in W such that $C \cap (Bm_CB)$ is dense in C. In the first part of the paper, we prove the following characterization of θ-twisted spherical conjugacy classes in G.

Theorem 1.1. *For $\theta \in \mathrm{Aut}'(G)$, a θ-twisted conjugacy class $C \subset G$ is spherical if and only if*

$$\dim C = l(m_C) + \mathrm{rk}(1 - m_C\theta). \tag{1.1}$$

When $\theta = \mathrm{Id}_G$, the identity automorphism of G, Theorem 1.1 is proved by N. Cantarini, G. Carnovale, and M. Costantini in [1] by a case-by-case checking that depends on the classification of all spherical conjugacy classes in G (for G simple). Formula (1.1) is then used in [1] to prove the De Concini-Kac-Procesi conjecture on representations of the quantized enveloping algebra of G at roots of unity over spherical conjugacy classes. A different proof of Theorem 1.1 for $\theta = \mathrm{Id}_G$, which is also valid when the characteristic of $\mathbf{k}$ is an odd good prime for G, is given by G. Carnovale in [2], where the proof does not require a classification of spherical conjugacy classes in G but it also depends on some case-by-case computations. When $\theta^2 = \mathrm{Id}_G$ and C is the θ-twisted conjugacy class through the identity element of G, (1.1) follows from standard results on symmetric spaces (see §2.3).

In §2, we give a direct proof of Theorem 1.1.

For $\theta = \mathrm{Id}_G$, the elements m_C play an important role in the study of spherical conjugacy classes. In particular, it is shown by M. Costantini [5] that the coordinate ring of a spherical conjugacy class C as a G-module is almost entirely determined by m_C (see [5, Theorem 3.22]). For G simple and of classical type and for $\theta = \mathrm{Id}_G$, the element m_C for every conjugacy class in G is computed explicitly in [4]. The second part of the paper concerns the set

$$\widetilde{\mathcal{M}}_\theta = \{m_C : \ C \text{ is a } \theta\text{-twisted conjugacy class in } G\} \subset W \tag{1.2}$$

for an arbitrary $\theta \in \mathrm{Aut}'(G)$. The set $\widetilde{\mathcal{M}}_\theta$ depends only on the isogeny class of G ([3, Remark 2]) and the automorphism of the Dynkin diagram of G induced by θ (Remark 1.2). Let

$$\begin{aligned}\mathcal{M}_\theta = \{m \in W : \ & m \text{ is the unique maximal length element} \\ & \text{in its } \theta\text{-twisted conjugacy class in } W\}.\end{aligned} \tag{1.3}$$

(See §3.1). By [3, Corollary 2.15], $\widetilde{\mathcal{M}}_\theta \subset \mathcal{M}_\theta$.

For G simple and θ an inner automorphism of G, it is shown in [3, §3] that $\widetilde{\mathcal{M}}_\theta = \mathcal{M}_\theta$ and elements in $\mathcal{M}_\theta$ are classified in [3, §3] using results from [1, 2]. For G simple and θ an outer automorphism of G, we prove in Theorem 3.8 that, again, $\widetilde{\mathcal{M}}_\theta = \mathcal{M}_\theta$, and we give in Proposition 3.7 the complete list of elements in $\mathcal{M}_\theta$. It turns out that if θ induces an order 2 automorphism of the Dynkin diagram, the list of elements in $\mathcal{M}_\theta$ coincides with that of T. A. Springer in [9, Table 2], and if $G = D_4$ and θ has order 3, $\mathcal{M}_\theta$ has two elements. The classification of elements in $\widetilde{\mathcal{M}}_\theta$ gives restrictions on the possible dimensions of θ-twisted conjugacy classes in G. See Example 3.9.

1.2. **Notation.** Let Δ_+ and $\Gamma \subset \Delta_+$ be the sets of positive and simple roots determined by (B, H) and write $\alpha > 0$ (resp. $\alpha < 0$) for $\alpha \in \Delta_+$ (resp. $\alpha \in -\Delta_+$). Let N and N_- be respectively the uniradicals of B and the opposite Borel subgroup B_-. The Lie algebras of G, B, H, N, and N_- are respectively denoted by $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{n}$, and $\mathfrak{n}_-$. For $\alpha \in \Delta_+$, s_α denotes the corresponding reflection in W. For each $w \in W$, we fix a representative $\dot{w}$ of w in $N_G(H)$.

For $\theta \in \mathrm{Aut}'(G)$, we use the same letter to denote the action of θ on Δ_+, and when necessary, we write $\theta \in \mathrm{Aut}(\Gamma)$ to indicate that θ is regarded as an automorphism of the Dynkin diagram. The induced action of θ on $\mathfrak{g}$ is also denoted by θ.

For $g \in G$, Ad_g denotes both the conjugation on G by g and the induced map on $\mathfrak{g}$. For a set V and a map $\sigma : V \to V$, we let $V^\sigma = \{x \in V : \sigma(x) = x\}$.

Remark 1.2. For an arbitrary $\theta_1 \in \mathrm{Aut}(G)$, there exists $g_0 \in G$ such that $\mathrm{Ad}_{g_0}(B) = \theta_1(B)$ and $\mathrm{Ad}_{g_0}(H) = \theta_1(H)$, so $\theta = \mathrm{Ad}_{g_0}^{-1} \circ \theta_1 \in \mathrm{Aut}'(G)$, and the right translation by g_0 in G maps θ_1-twisted conjugacy classes in G to θ-twisted conjugacy classes in G. We can thus assume throughout the paper that $\theta \in \mathrm{Aut}'(G)$. Moreover, if θ and $\theta' \in \mathrm{Aut}'(G)$ are in the same inner class, i.e., if they induce the same automorphism on the Dynkin diagram, then $\theta = \mathrm{Ad}_h \circ \theta'$ for some $h \in H$, and it follows that $\widetilde{\mathcal{M}}_\theta = \widetilde{\mathcal{M}}_{\theta'}$.

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2. Proof of Theorem 1.1

2.1. **Two lemmas on B-orbits in G.** Recall that $\cdot_\theta$ denotes the θ-twisted conjugacy action of G on itself. For $g \in G$, let B_g be the stabilizer subgroup of B at g. The following generalization of [1, Theorem 5] is proved in [6, Theorem 4.1]. We include the (short) proof for the convenience of the reader and to make the proof of Theorem 1.1 self-contained.

Lemma 2.1. [6] *For any $w \in W$ and $g \in wB$, one has $B_g \subset H^{w\theta}(N \cap \mathrm{Ad}_{\dot{w}}(N))$. Consequently,*

$$\dim B \cdot_\theta g \geq l(w) + \mathrm{rk}(1 - w\theta).$$

Proof. Let $b = n_1 n_2 h \in B_g$, where $h \in H$, $n_1 \in N \cap \mathrm{Ad}_{\dot{w}}(N_-)$ and $n_2 \in N \cap \mathrm{Ad}_{\dot{w}}(N)$. It follows from $bg = g\theta(b)$ and the unique decomposition $BwB = (N \cap \mathrm{Ad}_{\dot{w}}(N_-))\dot{w}B$ that $n_1 = 1$ and $w\theta(h) = h$. Thus $B_g \subset H^{w\theta}(N \cap \mathrm{Ad}_{\dot{w}}(N))$, and

$$\begin{aligned}\dim B \cdot_\theta g = \dim B - \dim B_g &\geq \dim B - \dim(N \cap \mathrm{Ad}_{\dot{w}}(N)) - \dim H^{w\theta} \\ &= l(w) + \mathrm{rk}(1 - w\theta).\end{aligned}$$

Q.E.D.

Lemma 2.2. *If $w \in W$ and $g \in wB$ are such that $B \cdot_\theta g$ is open in $G \cdot_\theta g$, then B_g is an open subgroup of $H^{w\theta}(N \cap \mathrm{Ad}_{\dot{w}}(N))$.*

Proof. Let $\mathfrak{g}_g = \{x \in \mathfrak{g} : \mathrm{Ad}_g\theta(x) = x\}$ be the stabilizer subalgebra of $\mathfrak{g}$ at g for the θ-twisted conjugation action, and let $\mathfrak{b}_g = \mathfrak{b} \cap \mathfrak{g}_g$. By Lemma 2.1, $\mathfrak{b}_g \subset \mathfrak{h}^{w\theta} + \mathfrak{n} \cap \mathrm{Ad}_{\dot{w}}(\mathfrak{n})$. It remains to prove that $\mathfrak{h}^{w\theta} + \mathfrak{n} \cap \mathrm{Ad}_{\dot{w}}(\mathfrak{n}) \subset \mathfrak{b}_g$.

Let $x_0 \in \mathfrak{h}^{w\theta}$ and $x_+ \in \mathfrak{n} \cap \mathrm{Ad}_{\dot{w}}(\mathfrak{n})$, and let $z = (\mathrm{Ad}_g\theta)^{-1}(x_+ + x_0) - (x_+ + x_0)$ so that $\mathrm{Ad}_g\theta(z + x_+ + x_0) = x_+ + x_0$. Using the fact that $\mathrm{Ad}_b(x_0) - x_0 \in \mathfrak{n}$ for any $b \in B$, one sees that $z \in \mathfrak{n}$. We now show that $z = 0$. To this end, let $\langle \, , \rangle$ be the Killing form of $\mathfrak{g}$. Since $B \cdot_\theta g$ is open in $G \cdot_\theta g$, the inclusion $\mathfrak{b} \hookrightarrow \mathfrak{g}$ induces an isomorphism $\mathfrak{b}/\mathfrak{b}_g \cong \mathfrak{g}/\mathfrak{g}_g$. Thus for any $y \in \mathfrak{g}$, there exists $y' \in \mathfrak{b}$ such that $y - y' \in \mathfrak{g}_g$, and, using $\langle z, \, y' \rangle = 0$, one has

$$\begin{aligned}\langle z, \, y \rangle &= \langle z + x_+ + x_0, \, y - y' \rangle - \langle x_+ + x_0, \, y - y' \rangle \\ &= \langle z + x_+ + x_0, \, y - y' \rangle - \langle \mathrm{Ad}_g\theta(z + x_+ + x_0), \, \mathrm{Ad}_g\theta(y - y') \rangle = 0.\end{aligned}$$

It follows that $z = 0$ and hence $x_+ + x_0 \in \mathfrak{b}_g$. Therefore $\mathfrak{b}_g = \mathfrak{h}^{w\theta} + \mathfrak{n} \cap \mathrm{Ad}_{\dot{w}}(\mathfrak{n})$.

Q.E.D.

2.2. **Proof of Theorem 1.1.** Let C be a θ-twisted conjugacy class in G. Assume first that $\dim C = l(m_C) + \mathrm{rk}(1 - m_C\theta)$. By Lemma 2.1, every B-orbit in $C \cap (Bm_CB)$ is open in C, so C is spherical.

Assume that C is spherical. Let $g \in C$ be such that $B \cdot_\theta g$ is open in C, and let $g \in BwB$ with $w \in W$. Then $C \cap (BwB) \supset B \cdot_\theta g$ is dense in C, so $w = m_C$. By Lemma 2.2,

$$\dim C = \dim \mathfrak{b} - \dim \mathfrak{b}_g = l(m_C) + \mathrm{rk}(1 - m_C\theta).$$

This finishes the proof of Theorem 1.1.

Remark 2.3. For $\theta = \mathrm{Id}_G$, Lemma 2.2 is also proved in [2] by some case-by-case arguments. On the other hand, the arguments in [2] are valid when the characteristic of $\mathbf{k}$ is an odd good prime for G, while our proof of Lemma 2.2 is valid when the Killing for of $\mathfrak{g}$ is non-degenerate and when one has the identifications of tangent spaces $T_g(B \cdot_\theta g) \cong \mathfrak{b}/\mathfrak{b}_g$ and $T_g(G \cdot_\theta g) \cong \mathfrak{g}/\mathfrak{g}_g$, which hold when $\mathbf{k}$ is of characteristic zero.

2.3. **The case of symmetric spaces.** Assume that $\theta \in \mathrm{Aut}'(G)$ is an involution, and let $K = G^\theta$ be the fixed point subgroup of θ in G. Then the θ-twisted conjugacy class C of the identity element of G is isomorphic to the symmetric space G/K, and it is well-known [8] that G/K is spherical. In this case, formula (1.1) for the dimension of G/K follows from results in [8]. Indeed, using the notation in [8, §5], let v^o be the unique open B-orbit in G/K and let $w^o = \phi(v^o) \in W$. Then $w^o = m_C$, and it is easy to see from [8, Corollary 4.9] that $\dim G/K = \frac{1}{2}\mathrm{Card}(C''_{v^o}) + \mathrm{Card}(I^n_{v^o}) + l(w^o) + \mathrm{rk}(1 - w^o\theta)$, where the notation is as on [8, Page 535]. By [8, Theorem 5.2(i)], $C''_{v^o} \cap \Gamma = \emptyset$. For every $\beta > 0$, writing $\beta = \beta_1 + \beta_2$, where β_1 is in the linear span of $\Pi \subset \Gamma$ in the notation of [8, Theorem 5.2(ii)] and β_2 is in the linear span of $\Gamma \backslash \Pi$, one has $w^o\theta(\beta) = \beta_1 + w^o\theta(\beta_2)$, so by [8, Theorem 5.2(ii)], $w^o\theta(\beta) > 0$ implies that $\beta_2 = 0$ and thus $w^o\theta(\beta) = \beta$. This shows that $C''_{v^o} = \emptyset$ and that every $\beta \in I_{v^o}$ is in the linear span of Π, which, by [8, Theorem 5.2(i)], consists of all simple compact imaginary roots. It follows that $I^n_{v^o} = \emptyset$. Thus $\dim G/K = l(w^o) + \mathrm{rk}(1 - w^o\theta)$.

3. The elements m_C

3.1. **Properties of $m \in \mathcal{M}_\theta$.** Any $\delta \in \mathrm{Aut}(\Gamma)$ induces an automorphism on the Weyl group W, also denoted by δ, by $\delta(w) = \delta \circ w \circ \delta^{-1} : \mathfrak{h} \to \mathfrak{h}$. Define the δ-twisted conjugacy of W on itself by $w \cdot_\delta v = wv\delta(w)^{-1}$ for $w, v \in W$ and call its orbits δ-twisted conjugacy classes in W. Let w_0 be the longest element in W, and let δ_0 be the automorphism of Γ given by $\delta_0(\alpha) = -\alpha$ for $\alpha \in \Gamma$. The automorphism on W induced by δ_0 is then given by $\delta_0(w) = w_0ww_0$ for $w \in W$.

Throughout this section, $\theta \in \mathrm{Aut}(\Gamma)$, and $\mathcal{M}_\theta \subset W$ is defined as in (1.3).

Lemma 3.1. *If $m \in \mathcal{M}_\theta$, then $\theta(m) = \delta_0(m) = m$.*

Proof. Let $m \in \mathcal{M}_\theta$. Then $\theta(m) = m^{-1}m\theta(m)$ is in the same θ-twisted conjugacy class as m, and $l(\theta(m)) = l(m)$. Thus $\theta(m) = m$. Similarly, since θ permutes the simple roots, $\theta(w_0) = w_0$. It follows that w_0mw_0 and m are in the same θ-twisted conjugacy class in W. Since $l(w_0mw_0) = l(m)$, one has $w_0mw_0 = m$.

Q.E.D.

For $I \subset \Gamma$, let $w_{0,I}$ be the longest element in the subgroup W_I of W generated by I.

The following Lemma 3.2 is proved in [3, §3] when θ is the identity automorphism of Γ.

Lemma 3.2. *If $m \in \mathcal{M}_\theta$, then $w_0m = mw_0 = w_{0,I}$, where $I = \{\alpha \in \Gamma : m\theta(\alpha) = \alpha\}$. In particular, I is both δ_0 and θ invariant, and $\delta_0\theta(\alpha) = -w_{0,I}(\alpha)$ for every $\alpha \in I$.*

Proof. Let $\delta = \delta_0\theta \in \mathrm{Aut}(\Gamma)$. Then the map $W \to W : w \mapsto ww_0$ maps θ-twisted conjugacy classes in W to δ-twisted conjugacy classes in W.

Let $m \in \mathcal{M}_\theta$, and let $x = mw_0$. Then x is a unique minimal length element in its δ-twisted conjugacy class in W. Let $x = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k}$ be a reduced word for x. Let $I' = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Then $x \in W_{I'}$. We first show that $x = w_{0,I'}$. To this end, it is enough to show that $x(\alpha_j) < 0$ for every $1 \leq j \leq k$. Since $xs_{\alpha_k} < x$, we already know that $x(\alpha_k) < 0$. If $k = 1$, we are done. Suppose that $k \geq 2$. Let $\beta_k = \delta^{-1}(\alpha_k) \in \Gamma$, and let

$$x_1 = s_{\beta_k}x\delta(s_{\beta_k}) = s_{\beta_k}xs_{\alpha_k} = s_{\beta_k}s_{\alpha_1}\cdots s_{\alpha_{k-1}}. \tag{3.1}$$

Since k is the minimal length of elements in the δ-twisted conjugacy class of x in W, we have $l(x_1) \geq k$. It follows from (3.1) that $l(x_1) \leq k$, so $l(x_1) = k$. Since x is the unique element in its δ-twisted conjugacy class in W with length k, we have $x_1 = x$. In particular, $x = x_1 = s_{\beta_k}s_{\alpha_1}\cdots s_{\alpha_{k-1}}$ is a reduced word for x, so $x(\alpha_{k-1}) < 0$. Repeating this process, we see that $x(\alpha_j) < 0$ for every $1 \leq j \leq k$. Thus $x = w_0m = mw_0 = w_{0,I'}$. It follows from Lemma 3.1 that $\delta_0(I') = \theta(I') = I'$.

We now show that $I' = I$. For any $\alpha \in I'$, since $m(\alpha) = w_0w_{0,I'}(\alpha) > 0$, one has $l(\theta^{-1}(s_\alpha)ms_\alpha) \geq l(m)$. Since $m \in \mathcal{M}_\theta$, one has $\theta^{-1}(s_\alpha)ms_\alpha = m$, so $\theta^{-1}(\alpha) = m(\alpha)$ and $\alpha \in I$. Conversely, let $\alpha \in I$. If $\alpha \notin I'$, then $w_0m(\alpha) = w_{0,I'}(\alpha) > 0$, so $m(\alpha) < 0$, contradicting the fact that $m(\alpha) = \theta^{-1}(\alpha) > 0$. Thus $I' = I$. It follows from the definition of I that $\delta_0\theta(\alpha) = -w_{0,I}(\alpha)$ for every $\alpha \in I$.

Q.E.D.

An element $w \in W$ is said to be a θ-twisted involution if $\theta(w) = w^{-1}$.

Corollary 3.3. *Every $m \in \mathcal{M}_\theta$ is both an involution and a θ-twisted involution.*

Proof. Let $m \in \mathcal{M}_\theta$ and let the notation be as in Lemma 3.2. Then $m^2 = w_0w_{0,I}w_{0,I}w_0 = 1$. Since $\theta(m) = m$, one also has $\theta(m) = m^{-1}$.

Q.E.D.

Definition 3.4. A subset I of Γ is said to have Property (1) if I is both δ_0 and θ invariant and if $\delta_0\theta(\alpha) = -w_{0,I}(\alpha)$ for all $\alpha \in I$.

By Lemma 3.2, every $m \in \mathcal{M}_\theta$ is of the form $m = w_0w_{0,I}$ for some $I \subset \Gamma$ with Property (1). The following Definition 3.5 is inspired by [2, Lemma 4.1].

Definition 3.5. For a subset I of Γ, an $\alpha \in I$ is said to be isolated if $\langle\alpha, \alpha'\rangle = 0$ for every $\alpha' \in I\backslash\{\alpha\}$. A subset I of Γ is said to have Property (2) if for every isolated $\alpha \in I$, there is no $\beta \in \Gamma\backslash\{\alpha\}$ with the following properties

(a) $\langle\alpha, \alpha\rangle = \langle\beta, \beta\rangle$ and $\langle\beta, \alpha\rangle \neq 0$;

(b) $\langle\beta, \alpha'\rangle = 0$ for all $\alpha' \in I\backslash\{\alpha\}$;

(c) $\delta_0\theta(\beta) = \beta$.

Lemma 3.6. *For every $m \in \mathcal{M}_\theta$, $I_m = \{\alpha \in \Gamma : m\theta(\alpha) = \alpha\} \subset \Gamma$ has Property (2).*

Proof. Let $m \in \mathcal{M}_\theta$. Suppose that $\alpha \in I_m$ is isolated and that there exists $\beta \in \Gamma\backslash\{\alpha\}$ with properties (a), (b), and (c) in Definition 3.5. Let $I'_m = I_m\backslash\{\alpha\}$. Since $\alpha \in I_m$ is isolated, $w_{0,I_m} = s_\alpha w_{0,I'_m}$, so by (b) and (c), $m\theta(\beta) = w_{0,I_m}w_0\theta(\beta) = -s_\alpha w_{0,I'_m}(\beta) = -s_\alpha(\beta)$, and

$$s_\alpha s_\beta m s_{\theta(\beta)} s_{\theta(\alpha)} = s_\alpha s_\beta s_{m\theta(\beta)} m s_{\theta(\alpha)} = s_\alpha s_\beta s_\alpha s_\beta s_\alpha m s_{\theta(\alpha)}.$$

By (a), $s_\alpha s_\beta s_\alpha s_\beta s_\alpha = s_\beta$, so $s_\alpha s_\beta m s_{\theta(\beta)} s_{\theta(\alpha)} = s_\beta m s_{\theta(\alpha)} = s_\beta s_\alpha m$. Since $m^{-1}(\alpha) = \theta(\alpha) > 0$, $l(s_\beta s_\alpha m) \geq l(m)$. Since $s_\alpha s_\beta m$ is in the same θ-twisted conjugacy class as m, we have $s_\beta s_\alpha m = m$, or $s_\alpha s_\beta = 1$, which is a contradiction.

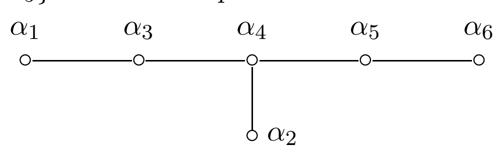
Q.E.D.

3.2. **The classification of $m \in \mathcal{M}_\theta$.** For $\theta \in \mathrm{Aut}(\Gamma)$, let $\mathcal{I}_\theta$ be the collection of all subsets I of Γ that have Properties (1) and (2). Note that the empty set $\emptyset$ is always in $\mathcal{I}_\theta$. Also note that if $\theta \in \mathrm{Aut}(\Gamma)$ is not the identity automorphism, then Γ does not have Property (1), so $\Gamma \notin \mathcal{I}_\theta$.

Proposition 3.7. *1) For $G = D_4$ and $\theta \in \mathrm{Aut}(\Gamma)$ of order 3, $I \in \mathcal{I}_\theta$ if and only if $I = \emptyset$ or $I = \{\alpha_2\}$, where α_2 is the simple root that is not orthogonal to any of the other three.*

2) For G simple and $\theta \in \mathrm{Aut}(\Gamma)$ of order 2, the list for $I \in \mathcal{I}_\theta$ is the same as that given in [9, Table 2]*, namely, either I is the empty set or I is one the following:*

A_{2n}, $n \geq 1$, $\theta = \delta_0$: no non-empty I in $\mathcal{I}_\theta$.
A_{2n+1}, $n \geq 1$, $\theta = \delta_0$: $I = \{\alpha_{2l+1} : 0 \leq l \leq n\}$.
D_4 : $I = \{\alpha_2\} \cup \Gamma(2,\theta)$, where $\Gamma(2,\theta)$ is the θ-orbit in Γ with 2 elements.
D_{2n}, $n > 2$, $\theta(\alpha_{2n-1}) = \alpha_{2n}$: $I_l = \Gamma\backslash\{\alpha_1, \alpha_2, \ldots, \alpha_{2l-1}\}$ for $1 \leq l \leq n-1$.
D_{2n+1}, $n \geq 2$, $\theta = \delta_0$: $I_l = \Gamma\backslash\{\alpha_1, \alpha_2, \ldots, \alpha_{2l-1}\}$ for $1 \leq l \leq n$.
E_6, $\theta = \delta_0$: $I = \{\alpha_3, \alpha_4, \alpha_5\}$ with the simple roots labeled as



Proof. 1) is easy to deduce and 2) is proved case-by-case. We omit the details.

Q.E.D.

By Lemma 3.2 and Lemma 3.6, we have the well-defined map

$$\psi : \quad \mathcal{M}_\theta \longrightarrow \mathcal{I}_\theta : \quad m \longmapsto I_m = \{\alpha \in \Gamma : m\theta(\alpha) = \alpha\}.$$

Since $m = w_0 w_{0,I_m}$ for every $m \in \mathcal{M}_\theta$, the map ψ is injective.

Assume now $\theta \in \mathrm{Aut}'(G)$, and let $\widetilde{\mathcal{M}}_\theta \subset W$ be defined as in (1.2). By Remark 1.2, $\widetilde{\mathcal{M}}_\theta$ depends only on the corresponding $\theta \in \mathrm{Aut}(\Gamma)$. Let $\widetilde{\psi} : \widetilde{\mathcal{M}}_\theta \to \mathcal{I}_\theta$ be the restriction of ψ to $\widetilde{\mathcal{M}}_\theta \subset \mathcal{M}_\theta$.

Theorem 3.8. *For G simple and $\theta \in \mathrm{Aut}'(G)$ an outer automorphism of G, the map $\widetilde{\psi} : \widetilde{\mathcal{M}}_\theta \to \mathcal{I}_\theta$ is bijective. Consequently,*

$$\widetilde{\mathcal{M}}_\theta = \mathcal{M}_\theta = \{w_0 w_{0,I} : I \in \mathcal{I}_\theta\}.$$

Proof. It is enough to prove that $\tilde{\psi}$ is surjective, and we may assume that G is adjoint.

First assume that $\theta \in \mathrm{Aut}(\Gamma)$ has order 2, and let $I \in \mathcal{I}_\theta$. By Proposition 3.7, I is in [9, Table 2], so $(I, \delta_0\theta)$ is *admissible* in the sense of [9, No. 2.2]. By [9, No. 4 and No. 5], there exists $h \in H$ such that $\mathrm{Ad}_h\theta \in \mathrm{Aut}(G)$ is an involution and that $w_0 w_{0,I} = m_C$, where C is the θ-twisted conjugacy class through h. In particular, $w_0 w_{0,I} \in \widetilde{\mathcal{M}}_\theta$.

It remains to consider the case of $G = D_4$ with $\theta \in \mathrm{Aut}(\Gamma)$ having order 3. It is clear that $w_0 = m_C$ if C is the θ-twisted conjugacy class of $\dot{w}_0$, so $w_0 \in \widetilde{\mathcal{M}}_\theta$. We only need to show that $w_0 s_2 \in \widetilde{\mathcal{M}}_\theta$. To this end, we may, by Remark 1.2, assume that $\theta \in \mathrm{Aut}'(G)$ is a diagram automorphism of G in the sense that $\theta \circ x_\alpha = x_{\theta\alpha}$ for $\alpha \in \Gamma$, where for each $\alpha \in \Gamma$, $x_\alpha : \mathbf{k}_a \to G$ is a fixed choice of one-parameter root subgroup corresponding to α. In particular, $\theta^3 = \mathrm{Id}_G$. Let C_e be the θ-twisted conjugacy class through the identity element e of G. It is well-known that $\mathfrak{g}^\theta$ is of type G_2 [7, Chapter 24] so it is 14-dimensional. Thus

$$\dim C_e = \dim G - 14 = 14 = l(w_0 s_2) + \mathrm{rk}(1 - w_0 s_2 \theta).$$

Since $l(w_0) + \mathrm{rk}(1 - w_0\theta) = 16$, we know by Lemma 2.1 that $m_{C_e} \neq w_0$ so $m_{C_e} = w_0 s_2$. In particular, $w_0 s_2 \in \widetilde{\mathcal{M}}_\theta$ and C_e is spherical. See [6, §4.5] for another proof of the fact that $w_0 s_2 \in \widetilde{\mathcal{M}}_\theta$ and that C_e is spherical.

Q.E.D.

Example 3.9. Let $G = D_4$ be of adjoint type, and let $\theta \in \mathrm{Aut}'(G)$ be a triality automorphism of G as in the proof of Theorem 3.8. Since $l(w_0 s_2) + \mathrm{rk}(1 - w_0 s_2\theta) = 14$ and $l(w_0) + \mathrm{rk}(1 - w_0\theta) = 16$, $\dim C \geq 14$ for every θ-twisted conjugacy class C in G, and, by Theorem 1.1, $\dim C = 14$ or 16 if C is spherical.

Recall from [10] that a θ-twisted conjugacy class is semisimple if it contains an element in H. For $h \in H$, let $C_h \subset G$ be the θ-twisted conjugacy class of h. Label the simple roots as $\Gamma = \{\alpha_j : 1 \leq j \leq 4\}$ such that $\theta(\alpha_2) = \alpha_2$, $\theta(\alpha_1) = \alpha_3, \theta(\alpha_3) = \alpha_4$, and $\theta(\alpha_4) = \alpha_1$. We now show that if $h^{\alpha_2} = h^{\alpha_1} h^{\alpha_3} h^{\alpha_4} = 1$, then $m_{C_h} = w_0 s_2$ and C_h is spherical, and otherwise, $m_{C_h} = w_0$ and $\dim C_h \geq 20$, so C_h is not spherical. Here, for a character μ on H, h^μ denotes the value of μ on h.

Label the positive roots in $\Delta_+ \backslash \Gamma$ as

$$\alpha_5 = \alpha_1 + \alpha_2, \quad \alpha_6 = \alpha_2 + \alpha_3, \quad \alpha_7 = \alpha_2 + \alpha_4,$$
$$\alpha_8 = \alpha_1 + \alpha_2 + \alpha_3, \quad \alpha_9 = \alpha_2 + \alpha_3 + \alpha_4, \quad \alpha_{10} = \alpha_1 + \alpha_2 + \alpha_4,$$
$$\alpha_{11} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \alpha_{12} = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4.$$

Then $\{\alpha_1, \alpha_3, \alpha_4\}, \{\alpha_5, \alpha_6, \alpha_7\}$ and $\{\alpha_8, \alpha_9, \alpha_{10}\}$ are the three θ-orbits in Δ_+ of size 3 and $\theta(\alpha_{11}) = \alpha_{11}$ and $\theta(\alpha_{12}) = \alpha_{12}$. Note that the sets $\{\alpha_1, \alpha_3, \alpha_4, \alpha_{12}\}$, $\{\alpha_5, \alpha_6, \alpha_7, \alpha_{11}\}$, and $\{\alpha_8, \alpha_9, \alpha_{10}, \alpha_2\}$ consist of strongly orthogonal roots, and, with s_j denoting the reflection in W defined by α_j for $1 \leq j \leq 12$, $w_0 = s_1 s_3 s_4 s_{12} = s_5 s_6 s_7 s_{11} = s_8 s_9 s_{10} s_2$.

Recall that the stabilizer subalgebra of $\mathfrak{g}$ at h is $\mathfrak{g}_h = \mathfrak{g}^{\mathrm{Ad}_g\theta}$. Since $\dim \mathfrak{h}^{\mathrm{Ad}_h\theta} = \dim \mathfrak{h}^\theta = 2$, one has $\dim \mathfrak{g}_h = 2 + 2n$, where $n = \#\{i \in \{1, 2, 5, 8, 11, 12\} : \lambda_i(h) = 1\}$, with $\lambda_i(h) = h^{\alpha_i + \theta(\alpha_i) + \theta^2(\alpha_i)}$ for $i \in \{1, 5, 8\}$ and $\lambda_i(h) = h^{\alpha_i}$ for $i \in \{2, 11, 12\}$. Let $\Lambda(h) = \{\lambda_i(h) : i \in$

$\{1, 2, 5, 8, 11, 12\}\}$. Then $\lambda_1(h) = h^{\alpha_1} h^{\alpha_3} h^{\alpha_4}$, $\lambda_2(h) = h^{\alpha_2}$, and

$$\Lambda(h) = \{\lambda_1(h),\ \lambda_2(h),\ \lambda_1(h)(\lambda_2(h))^3,\ (\lambda_1(h))^2(\lambda_2(h))^3,\ \lambda_1(h)\lambda_2(h),\ \lambda_1(h)(\lambda_2(h))^2\}.$$

Case 1: $h^{\alpha_2} = h^{\alpha_1} h^{\alpha_3} h^{\alpha_4} = 1$. In this case, $n = 6$, $\dim \mathfrak{g}_h = 14$, and $\dim C_h = 28 - 14 = 14$. It follows from Lemma 2.1 that C_h is spherical and $m_{C_h} = w_0 s_2$. Note that in this case, $(\mathrm{Ad}_h \theta)^3 = \mathrm{Id}_G$, so $\mathfrak{g}_h$ is again of tyep G_2.

Case 2: $h^{\alpha_2} \neq 1$ or $h^{\alpha_1} h^{\alpha_3} h^{\alpha_4} \neq 1$. In this case, $n \leq 5$. In fact, it is easy to see that one can not have $n = 5$ nor $n = 4$, so $n \leq 3$, and $\dim C_h = 28 - \dim \mathfrak{g}_h \geq 20$. Thus C_h is not spherical. We use the approach in [6, §4.5] to prove that $m_{C_h} = w_0$. First assume that $h^{\alpha_2} \neq 1$. Fix a one-parameter root subgroup $x_\alpha : \mathbf{k}_a \to G$ for $\alpha \in -\{\alpha_8, \alpha_9, \alpha_{10}\}$ such that $\theta \circ x_\alpha = x_{\theta(\alpha)}$ for every $\alpha \in -\{\alpha_8, \alpha_9, \alpha_{10}\}$ (recall that $\theta^3 = \mathrm{Id}_G$). For $a, b, c, d \in \mathbf{k}\backslash\{0\}$, let $g = x_{-\alpha_2}(a) x_{-\alpha_8}(b) x_{-\alpha_9}(c) x_{-\alpha_{10}}(d) \in G$. Then

$$gh\theta(g)^{-1} = x_{-\alpha_2}(a - h^{-\alpha_2}a) x_{-\alpha_8}(b - h^{-\alpha_8}d) x_{-\alpha_9}(c - h^{-\alpha_9}b) x_{-\alpha_{10}}(d - h^{-\alpha_{10}}c).$$

Choosing a, b, c, d such that $a \neq 0, b - h^{-\alpha_8}d \neq 0, c - h^{-\alpha_9}b \neq 0$ and $d - h^{-\alpha_{10}}c \neq 0$, one has $gh\theta(g)^{-1} \in C_h \cap (Bw_0B) \cap B_-$, so $m_{C_h} = w_0$. If $h^{\alpha_2} = 1$, then $h^{\alpha_1} h^{\alpha_3} h^{\alpha_4} \neq 1$. In this case, $h^{\alpha_{11}} = h^{\alpha_{12}} \neq 1$. Using the fact $w_0 = s_5 s_6 s_7 s_{11}$ or the fact $w_0 = s_1 s_3 s_4 s_{12}$ and arguments similar to the above, one sees that $m_{C_h} = w_0$.

$\diamondsuit$

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Jiang-Hua Lu, Department of Mathematics, Hong Kong University, Pokfulam Rd., Hong Kong
E-mail address: jhlu@maths.hku.hk